Basics

Gas with mass density ρ, pressure P, temperature T, velocity $\vec{u} = (u_r, u_\theta, u_\phi)$ or $\vec{u} = (u_R, u_\phi, u_z)$.
Assume ideal equation of state: $P = \frac{k_\mathrm{B}}{\mu m_\mathrm{H}}\rho T = \frac{\mathcal{R}}{\mu}\rho T$. Then, the internal energy density is $e = \rho c_\mathrm{V} T = \frac{P}{\gamma - 1}$.

Euler equations of hydrodynamics:

$$\frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho = -\rho(\nabla\cdot\vec{u}) \tag{1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\frac{1}{\rho}\nabla P - \nabla\Phi_\star \tag{2}$$

$$\frac{\partial e}{\partial t} + \vec{u}\cdot\nabla e = -\gamma e(\nabla\cdot\vec{u}) \tag{3}$$

The equation of state defines the *sound speed* c_s:

$$c_\mathrm{s} = \sqrt{\gamma\frac{\partial P}{\partial \rho}} \overset{\text{ideal}}{\Longrightarrow} c_\mathrm{s} = \sqrt{\gamma\frac{P}{\rho}} = \sqrt{\gamma\frac{\mathcal{R}T}{\mu}} \propto \sqrt{T}$$

Isothermal gas: $\gamma = 1 \to c_\mathrm{siso} := c_\mathrm{s}/\gamma \Rightarrow$ $\boxed{P = c_\mathrm{siso}^2\rho.}$

Gravitational potential of the star: $\Phi_\star = -\frac{\mathrm{G}M_\star}{r} = -\frac{\mathrm{G}M_\star}{\sqrt{R^2+z^2}}$. Keplerian rotation defines the *Keplerian angular frequency* $\Omega_\mathrm{K} = \frac{2\pi}{T_\mathrm{K}}$:

$$\frac{T_\mathrm{K}^2}{R^3} = \frac{4\pi^2}{\mathrm{G}M_\star} \quad\Rightarrow\quad \boxed{\Omega_\mathrm{K} = \sqrt{\frac{\mathrm{G}M_\star}{R^3}}}$$

Vertical stratification: assuming vertically isothermal gas ($\partial T/\partial z = 0$), hydrostatic equilibrium defines a *pressure scale height* H:

$$(2) \overset{u=0}{\Longrightarrow} \frac{\partial P}{\partial z} = -\rho\frac{\partial\Phi_\star}{\partial z} \overset{z\ll R}{\Longrightarrow} \rho(R,z) \approx \rho_\mathrm{mid}(R)\exp\left(-\frac{z^2}{2H^2}\right), \qquad \boxed{H := \frac{c_\mathrm{siso}}{\Omega_\mathrm{K}}}$$

More accurately, the pressure-supported equilibrium state if $\rho_\mathrm{mid}(R) \propto (R/R_0)^p$, $T \propto (R/R_0)^q$ is exactly:

$$\rho(R,z) = \rho_\mathrm{mid}(R)\exp\left[\frac{1}{h^2}\left(\frac{R}{\sqrt{R^2+z^2}} - 1\right)\right], \qquad u_\phi(R,z) = R\Omega_\mathrm{K}\left[1 + (p+q)h^2 + q\left(1 - \frac{R}{\sqrt{R^2+z^2}}\right)\right]^{1/2}.$$

Disks are generally geometrically thin ($H \ll R$). Define the *aspect ratio* $\boxed{h := H/R}$ ≈ 0.03–0.1.

Now, define the *surface density* $\Sigma(R) = \int_{-\infty}^{\infty}\rho(R,z)\mathrm{d}z$ and the vertically integrated pressure $P_\mathrm{2D}(R) = c_\mathrm{siso}^2\Sigma$ and energy density $e_\mathrm{2D}(R) = \frac{P_\mathrm{2D}}{\gamma-1}$. The equations can be *integrated vertically*:

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = -\Sigma(\nabla\cdot\vec{u}) \qquad \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\frac{1}{\Sigma}\nabla P_\mathrm{2D} - \nabla\Phi_\star \qquad \frac{\mathrm{d}e_\mathrm{2D}}{\mathrm{d}t} = -\gamma e_\mathrm{2D}(\nabla\cdot\vec{u}), \qquad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\vec{u}\cdot\nabla). \tag{4}$$

This also means that $\boxed{\Sigma \approx \sqrt{2\pi}\rho_\mathrm{mid}H.}$

For simplicity, from now on assume power laws such that $\Sigma = \Sigma_0\left(\frac{R}{R_0}\right)^s$, $T = T_0\left(\frac{R}{R_0}\right)^q$, $P_\mathrm{2D} = P_0\left(\frac{R}{R_0}\right)^{s+q}$.

The radial profile of the azimuthal velocity $u_\phi(R)$ is defined in hydrodynamic equilibrium:

$$(4) \overset{u_R=0}{\Longrightarrow} \frac{u_\phi^2}{R} = \frac{1}{\Sigma}\frac{\partial P_\mathrm{2D}}{\partial R} + \frac{\partial\Phi_\star}{\partial R} \quad\Rightarrow\quad \boxed{u_\phi = u_\mathrm{K}\sqrt{1+(s+q)h^2}} \approx 0.997\,u_\mathrm{K}, \qquad \boxed{u_\mathrm{K} = R\Omega_\mathrm{K}}$$

- Simple irradiation model for a passive (externally heated) disk: $q \approx -1/2$, or $q = -3/7$ (see CG97).
- Typical density profiles motivated by observations: $s \approx -1$. Since $p = s - (q+3)/2$, $p \approx -2.25$.

Accretion & viscosity

The disk is slowly falling onto the star (accretion)—define the accretion rate $\dot{M}$ in steady state:

$$(4) \overset{\frac{\partial}{\partial t}=0}{\Longrightarrow} \Sigma R u_R = \text{const.} \quad\Rightarrow\quad \boxed{\dot{M} := -2\pi\Sigma R u_R}$$

To drive accretion we need angular momentum transport radially outwards. This can be modeled as a *kinematic viscosity* ν. In a steady state, $\boxed{\dot{M} \approx 3\pi\nu\Sigma}$ $\Rightarrow$ $\boxed{u_R(R) = -\frac{3}{2}\frac{\nu}{R}.}$

SS73 viscosity model: $\nu = \alpha c_\mathrm{s} H$, $\alpha \ll 1$. Note that α can vary with R or z!

Dust dynamics

Solid material particles (affectionately called "dust") coexist with the gas ($M_\mathrm{dust} \sim 0.01 M_\mathrm{gas}$ in ISM). Because the dust does not feel the gas pressure, it orbits at u_K and thus feels a headwind $u_\mathrm{K} - u_\phi$ and drifts radially towards regions of higher pressure (typically inwards).

The coupling between the dust and the gas is given by the *stopping time* t_s, or *Stokes number* $\mathrm{St} = t_\mathrm{s}\Omega_\mathrm{K}$. In the Epstein regime (up to ~cm size):

$$\boxed{\mathrm{St} = \sqrt{\frac{\pi}{8}}\frac{a_\mathrm{d}\tilde{\rho}_\mathrm{d}}{\rho H}} \overset{z=0}{\Longrightarrow} \boxed{\mathrm{St}_\mathrm{mid} = \frac{\pi}{2}\frac{a_\mathrm{d}\tilde{\rho}_\mathrm{d}}{\Sigma},}$$

where a_d is the grain radius and $\tilde{\rho}_\mathrm{d} \sim 1\,\mathrm{g/cm^3}$ is the dust material density. Small (large) St means the dust is well (poorly) coupled to the gas. The dust drifts radially with a speed

$$u_{R,\mathrm{d}} \approx \frac{1}{1+\mathrm{St}^2}\left(u_{R,\mathrm{g}} + \frac{\mathrm{St}}{\Sigma\Omega_\mathrm{K}}\frac{\mathrm{d}P}{\mathrm{d}R}\right) = \frac{u_{R,\mathrm{g}}}{1+\mathrm{St}^2} + \frac{1}{\mathrm{St}+\mathrm{St}^{-1}}\eta u_\mathrm{K}, \qquad \eta = \frac{\mathrm{d}\log P}{\mathrm{d}\log R}h^2$$

which is fastest when $\mathrm{St} \sim 1$, and typically much faster than the disk lifetime (so pressure traps are needed to save mm-sized dust from falling onto the star)!

In the absence of turbulence, the dust settles to the midplane of the disk over a timescale $t_\mathrm{settle} = \mathrm{St}^{-1}\Omega_\mathrm{K}^{-1}$. If the dust is kicked up vertically via turbulence, it settles to an equilibrium profile given by the balance between vertical settling and turbulent diffusion with $\bar{\nu} = \alpha c_\mathrm{s} H$ ($\partial\alpha/\partial z = 0$):

$$\rho_\mathrm{d}(R,z) \approx \rho_\mathrm{d,mid}(R)\exp\left[-\frac{\mathrm{St}-\mathrm{St}_\mathrm{mid}}{\alpha} - \frac{z^2}{2H^2}\right], \qquad \mathrm{St} = \mathrm{St}_\mathrm{mid}\frac{\rho_\mathrm{mid}}{\rho} \approx \mathrm{St}_\mathrm{mid}\exp\left(\frac{z^2}{2H^2}\right).$$

Notice the degeneracy between α and St (always α/St)!

Radiative cooling

The disk cools over a cooling timescale $t_\mathrm{cool} = \beta\Omega_\mathrm{K}^{-1}$, where $\beta(\rho, T, \dots)$ is a dimensionless cooling parameter. To capture this locally (i.e., missing diffusive processes!), introduce a source term in Eq. (3):

$$\frac{\partial e}{\partial t} = Q_\mathrm{cool} = -\frac{\Delta e}{t_\mathrm{cool}} = -\frac{e - e_0}{t_\mathrm{cool}} \Rightarrow \boxed{\frac{\partial T}{\partial t} \approx -\frac{T - T_0}{\beta}\Omega_\mathrm{K},}$$

where T_0 is a floor temperature (e.g., $T_\mathrm{ISM} \approx 3\,\mathrm{K}$) or a reference profile. Examples of such profiles are disks in thermal equilibrium (e.g., an irradiation–emission balance) but are in principle heavily parameter-dependent or even arbitrary. Note that this really describes a *relaxation* mechanism, as for $T < T_0$ it *heats* the disk to T_0.

- In viscous α disks, $Q_\mathrm{visc} \approx \frac{9}{4}\nu\rho\Omega_\mathrm{K}^2$. Then thermal balance gives an equilibrium temperature T_eq:

$$\frac{\partial e}{\partial t} = Q_\mathrm{visc} + Q_\mathrm{cool} = 0 \Rightarrow \boxed{T_\mathrm{eq} = \frac{T_0}{1 - k\alpha\beta},} \qquad k := \frac{9}{4}\sqrt{\gamma}(\gamma - 1) \approx 1\text{–}2$$

- In case of an arbitrary cooling function, β can be approximated as:

$$\frac{\partial e}{\partial t} = Q_\mathrm{cool} \Rightarrow \frac{e}{t_\mathrm{cool}} \sim |Q_\mathrm{cool}| \Rightarrow t_\mathrm{cool} \approx \frac{e}{|Q_\mathrm{cool}|} \Rightarrow \boxed{\beta \approx \frac{e}{|Q_\mathrm{cool}|}\Omega_\mathrm{K}}$$

The disk can cool via thermal emission: gas exchanges heat with the (much more efficiently-cooling) dust, which then couples to the radiation field E_rad. Assuming an isotropic radiative flux $\vec{F}_\mathrm{rad}$ (*flux-limited diffusion* closure, or "FLD"), the thermal- and radiation energies evolve as

$$\frac{\partial E_\mathrm{rad}}{\partial t} = -\nabla\cdot\vec{F}_\mathrm{rad} + Q_\mathrm{rad}, \qquad \frac{\partial e}{\partial t} = -Q_\mathrm{rad}, \qquad \vec{F}_\mathrm{rad} = -\frac{\lambda c}{\rho\kappa_\mathrm{R}}\nabla E_\mathrm{rad}, \qquad Q_\mathrm{rad} = \kappa_\mathrm{P}\rho\left(a_\mathrm{R}T^4 - E_\mathrm{rad}\right),$$

where κ_R, κ_P are the Rosseland and Planck mean opacities, respectively, and λ is a flux limiter that handles the transition between the optically thick ($\lambda \to 1/3$, diffusion limit) and optically thin ($\vec{F}_\mathrm{rad} \to cE_\mathrm{rad}$, free-streaming limit) regimes. Many recipes for λ exist. A representative cooling timescale is:

$$\boxed{\beta \approx \beta_\mathrm{thick} + \beta_\mathrm{thin} \approx \frac{\Omega_\mathrm{K}}{\eta}\left(H^2 + \frac{l_\mathrm{rad}^2}{3}\right),} \qquad \eta = \frac{16\sigma_\mathrm{SB}T^3}{3\kappa_\mathrm{R}\rho^2 c_\mathrm{V}}, \qquad l_\mathrm{rad} = \frac{1}{\kappa_\mathrm{P}\rho},$$

Where κ_R and κ_P are Rosseland and Planck mean opacities, and we assumed that:

1. Gas–dust thermal coupling is much faster than β so that $T_\mathrm{g} \approx T_\mathrm{d}$ (not true for very low ρ_d).
2. Small perturbations from equilibrium so $E_\mathrm{rad} \approx a_\mathrm{R}T_0^4$ (not true in general).
3. The characteristic length scale for radiative diffusion in the optically thick limit is $\sim H$ (debatable).
4. The same length scale in the optically thin limit is the photon mean free path $\sim l_\mathrm{rad}$ (this is okay).
5. Cooling is dominated by tiny grains ($a_\mathrm{d} \sim \mu m$), perfectly coupled to the gas (St $\ll 1$).

Radiation is often numerically handled with β cooling (cheap, does not capture diffusion), or FLD (costly, does not capture shadows, works best in the opt. thick limit), or Monte Carlo techniques (very costly, works best in opt. thin limit). More sophisticated methods exist (M1, short characteristics, etc.).

References

Armitage, 2007 • Nelson et al., 2013 • Chiang & Goldreich, 1997 • Shakura & Sunyaev, 1973 • Fromang & Nelson, 2009 • Gammie, 2001 • Levermore & Pomraning, 1981 • Takeuchi & Lin, 2002

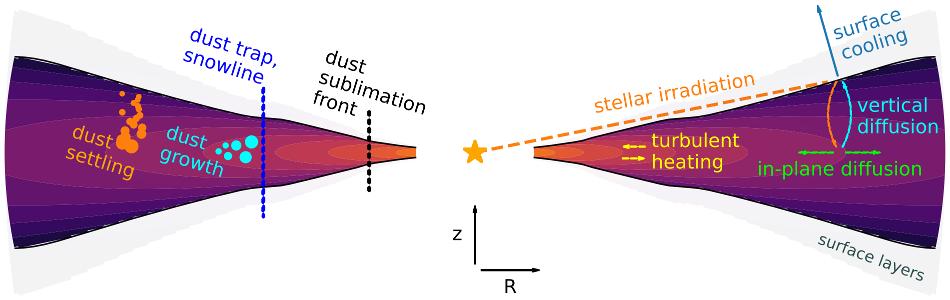
Miscellaneous stuff

Let $\Sigma \propto R^s$, $T \propto R^q$, $\nu = \alpha c_\mathrm{s} H$, then $Q \propto R^e$ where…

Q	e	Reminder
Σ	s	
T	q	
Ω_K	$-3/2$	Kepler's law
u_K	$-1/2$	$= R\,\Omega_\mathrm{K}$
c_siso, c_s	$q/2$	$\propto \sqrt{T}$
H	$\frac{q+3}{2}$	$= c_\mathrm{siso}/\Omega_\mathrm{K}$
h	$\frac{q+1}{2}$	$= H/R$
ρ_mid	$s - \frac{q+3}{2}$	$\propto \Sigma/H$
P	$p + q$	$\propto \rho T$
P_2D	$s + q$	$\propto \Sigma T$
ν	$q + 3/2$	$\propto c_\mathrm{s}H$, $\alpha =$ ct.
$\dot{M}$	$s + q + 3/2$	$\propto \nu\Sigma$

Handy conversion table between T-related quantities.

	T	H	h	c_siso
T	-	$\frac{\mu \mathrm{G}M_\star}{\mathcal{R}}\,\frac{H^2}{R^3}$	$\frac{\mu \mathrm{G}M_\star}{\mathcal{R}}\,\frac{h^2}{R}$	$\frac{\mu}{\mathcal{R}}\,c_\mathrm{siso}^2$
H	$\sqrt{\frac{\mathcal{R}}{\mu \mathrm{G}M_\star}}\,\sqrt{TR^3}$	-	$h\,R$	$c_\mathrm{siso}/\Omega_\mathrm{K}$
h	$\sqrt{\frac{\mathcal{R}}{\mu \mathrm{G}M_\star}}\,\sqrt{TR}$	H/R	-	$c_\mathrm{siso}/u_\mathrm{K}$
c_siso	$\sqrt{\mathcal{R}/\mu}\,\sqrt{T}$	$H\,\Omega_\mathrm{K}$	$h\,u_\mathrm{K}$	-



Sketch of dust (left) and thermal (right) processes in a protoplanetary disk, viewed edge-on.

Found an error, or want to see something added? Contact me at `a.ziampras@lmu.de`